Logic-based Tractable Approximations of Probability *

Paolo Baldi[1] and Hykel Hosni[1]

University of Milan,
{paolo.baldi,hykel.hosni}@unimi.it

Probabilities can be formulated as functions over sets or, equivalently, in logical terms, over formulas of classical logic [5]. The logical formulation helps in particular in highlighting two strong idealizations which result from the combination of classical logic and the axioms of probability. First, classical probability functions are unable to distinguish between "probabilistic knowledge" and "probabilistic ignorance", since $P(\neg\theta) = 1 - P(\theta)$ for any formula θ. This amounts to saying that they have no direct way of representing a very common situation: the agent doesn't known anything about θ. Another important problem comes from the monotonicity of probability functions with respect to $\vdash$: i.e.

$$\theta \vdash \varphi \text{ implies } P(\theta) \leq P(\varphi), \tag{1}$$

This property is mathematically convenient, but puts on an agent a very heavy burden which owes to the intractability of $\vdash$. Indeed, a logical deduction from θ to φ might be highly nontrivial and hard to find, making the application of (1) a constraint of rationality that realistic agents may not be in a position to comply with.

In our recent work [2] we tackle both problems by replacing classical logic $\vdash$ with the family of Depth-bounded Boolean logic (DBBL) investigated in [4, 3].

A characteristic feature of DBBLs is its informational nature. Whereas connectives in classical logic are defined in terms of truth-values, DBBL provide an informational view of logical consequence which, as a by-product, also provides a tractable approximation of it.

The central idea behind the (semantic) approach to DBBL is to distinguish two kinds of information. The first is information which the agent possesses explicitly or can trivially infer from it. For definiteness, it is the kind of information that an agent holds when she holds the information that a conjunction is true, i.e. that both conjuncts are true.

The second is information that the agent does not actually possess, but temporarily assumes in the course of *hypotethical reasoning*. This is, for instance, the information used in a proof by cases of a mathematical theorem.

The hierarchy of Depth-bounded Boolean logic arises by bounding the number k of allowed nested iterations of inferences using hypothetical information. If $k = 0$, then only information actually held by the agent can be used as the

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premises of a logical deduction, yielding the 0-depth consequence relation denoted by $\vdash_0$.

For $0 \leq k < m$ it can be shown that $\vdash_k \subset \vdash_m$, whereas $\lim_{k\to\infty} \vdash_k = \vdash$. This justifies interpreting the hierarchy of DBBLs as an approximation to classical logic, which is indeed attained when the agent is allowed an unbounded use of hypothetical information.

In analogy to DBBLs, we introduce a hierarchy of *Depth-bounded Belief functions* which approximate probability functions and asymptotically coincide with them. Our construction is inspired by the theory of Dempster-Shafer Belief functions [6], as suggested by our choice of terminology. As in DS-theory, none of the belief functions in our hierarchy is constrained by additivity, except for the one attained in the limit, which is a probability function.

Let us recall that Belief functions allow for $B(\varphi \vee \psi) \geq B(\varphi) + B(\psi)$, with $\varphi, \psi \vdash \bot$, since they reflect the information that an agent possesses. Information for a disjunction may indeed be held in the absence of any information about the disjuncts, as in the famous Ellsberg-like scenarios.

Our framework connects higher logical abilities of an agent (as captured by the index of the relation $\vdash_k$) with the ability to obtain increasingly tighter approximations of $B_k(\varphi)+B_k(\psi)$ by $B_k(\varphi\vee\psi)$. This puts forward a seemingly novel approach, a logic-based one, to Ellsberg-like scenarios. As a welcome byproduct, reasoning tasks, such as variants of PSAT, based on members B_k of our hierarchy, will turn out to be computationally tractable.

Our main results read as follows. We show that each probability function can be approximated by a hierarchy of Depth-bounded Belief functions, and, conversely, we single out the conditions under which our Depth-bounded Belief functions actually determine a probability in the limit. Finally, we prove that under rather palatable restrictions, the depth-bounded functions introduced here are an adequate tool to tackle the well-known unfeasibility of logic-based uncertain reasoning [5].

The framework and results presented here are based on [2]. If time allows, we will sketch ongoing work, first reported in [1], which provide an approximation of probabilities using *qualitative* depth-bounded belief functions.

References

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